Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.31, No. 3, 277-285

Topologically mixing and minimal but not ergodic, analytic transformation on $\mathbf{T}^5$

Bassam R. Fayad

Abstract. We give an example of an analytic transformation on $\mathbf{T}^5$ that conserves the Haar measure, that is minimal and topologically mixing, but is not ergodic.

Keywords: measure preserving, minimal, topologically mixing, nonergodic, time change, reparametrization.

1 Introduction

In [2], Furstenberg constructed an analytic diffeomorphism of $\mathbf{T}^2$ that preserves the Haar measure and is minimal but not ergodic. The diffeomorphism F he produces is not topologically mixing since there exists a sequence of integers $k_n \to \infty$ such that $F^{k_n} \to Id_{\mathbf{T}^2}$ uniformly as n goes to infinity (this *rigidity* obviously eliminates topological mixing). We will use the construction of Furstenberg and the techniques developed in [1] of reparametrizations of irrational flows on the torus in dimension higher than 3, to construct an example on $\mathbf{T}^5$ of a diffeomorphism that has all the properties of the Furstenberg map but that is in addition topologically mixing.

An essential ingredient of our construction will be the construction by J-C. Yoccoz in an appendix to his thesis [4], of a minimal translation on $\mathbf{T}^2$ and a real-analytic complex function φ of $\mathbf{T}^2$ that give a counterexample to the Denjoy-Koksma inequality in dimension 2. Following [4], we take α and α' rationally independent such that the denominators of their convergents, q_n and q'_n, satisfy

Received 9 March 2000.

for $n \geq n_0$

$$q_n \geq e^{3q'_{n-1}}, \tag{1}$$
$$q'_n \geq e^{3q_n}. \tag{2}$$

Define then

$$\varphi(x, y) = 1 + Re\left(\sum_{n=n_0}^{\infty} \frac{e^{i2\pi q_n x}}{e^{q_n}}\right) + Re\left(\sum_{n=n_0}^{\infty} \frac{e^{i2\pi q'_n y}}{e^{q'_n}}\right).$$

Assume n_0 is such that $\frac{1}{2} \leq \varphi(x, y) \leq \frac{3}{2}$, for any $(x, y) \in \mathbf{T}^2$. We will denote the Birkhoff sums of φ with respect to $R_{\alpha,\alpha'}$ by

$$\varphi_m(x, y) := \sum_{k=0}^{m-1} \varphi\left(R^k_{\alpha,\alpha'}(x, y)\right).$$

The *stretching* (important partial derivatives) of the Birkhoff sums φ_m for *all* large m will be central for topological mixing as we will explain later. For the moment we just state the only property of the sums φ_m that we will need:

Proposition 1 (Stretch). *Let be given a rectangle R on $\mathbf{T}^2$. There exists an interval $J \times \{y_0\} \subset R$ of length more than $1/{q_n}^2$, such that for any $m \in [\frac{e^{2q_n}}{2}, 2e^{2q'_n}]$ and any $x \in J$, we have*

$$\frac{\partial \varphi_m}{\partial x}(x, y_0) \geq \frac{m}{e^{q_n}}. \tag{3}$$

A similar statement involving $\frac{\partial \varphi_m}{\partial y}(x_0, y)$ holds for $m \in [\frac{e^{2q'_n}}{2}, 2e^{2q_{n+1}}]$.

This proposition follows from a direct computation of the φ_m's, and its proof can be found in [1] [1] or implicitly in [4]. The essential thing to notice is that the

[1] The exact statement in [1] is: Define, for $n \in \mathbf{N}$, the set

$$I_n = \{x \in \mathbf{T}^1 \;/\; [q_n x] \in [\frac{1}{n}, \frac{1}{2} - \frac{1}{n}] \bigcup [\frac{1}{2} + \frac{1}{n}, 1 - \frac{1}{n}]\},$$

then we have the following

Proposition 3.4. *For any $y \in \mathbf{T}^1$, for any $x \in I_n$, for any $m \in [\frac{e^{2q_n}}{2}, 2e^{2q'_n}]$,*

$$\left|\frac{\partial \varphi_m}{\partial x}(x, y)\right| \geq \frac{m}{e^{q_n}} \frac{q_n}{n}. \tag{4}$$

A similar inequality on $\frac{\partial \varphi_m}{\partial y}(x, y)$ holds when $m \in [\frac{e^{2q'_n}}{2}, 2e^{2q_{n+1}}]$.

intervals $[\frac{e^{2q_n}}{2}, 2e^{2q'_n}]$, $[\frac{e^{2q'_n}}{2}, 2e^{2q_{n+1}}]$ cover $\mathbf{N}$ when n runs through the integers, hence the derivatives of φ_m will always be stretching either in one or in the other direction x and y (or in both).

As we mentioned, the other ingredient of our construction will be Furstenberg's example. Choose β an irrational number such that the translation on $\mathbf{T}^3$, $R_{\alpha,\alpha',\beta}$ be minimal, and such that the sequence of denominators of the convergents of β, $\tilde{q}_n$ satisfy for $n \geq n_0$

$$\tilde{q}_n \geq e^{\tilde{q}_{n-1}}.$$

Let ϕ be the following real analytic function on $\mathbf{T}^1$:

$$\phi(\theta) = \sum_{n=1}^{\infty} \frac{\sin 2\pi \tilde{q}_n \theta}{n \tilde{q}_{n+1}}.$$

Next, define on $\mathbf{T}^4$ the following skew product, denoted by T :

$$\begin{aligned} \mathbf{T}^4 &\to \mathbf{T}^4, \\ (x, y, \theta, z) &\to \big(x+\alpha, y+\alpha', \theta+\beta, z+\phi(\theta)\big). \end{aligned}$$

What will be relevant for T is the following

Proposition 2. *The diffeomorphism T is minimal and nonergodic.*

Proof. This proposition is due to Furstenberg [2], and follows from our choice of ϕ (*wild coboundary*). The idea is that if the equation

$$\psi(\theta) - \psi(\theta+\beta) = \phi(\theta), \tag{E}$$

admits a measurable solution ψ but does not admit a continuous one, then the skew product T is nonergodic but is minimal (Cf [2], or [3], Propositions 4.2.5 and 4.2.6). Here, the solution ψ of the equation (E) one finds using Fourier expansions is:

$$\psi(\theta) = Re\left(-i\sum_{n=1}^{\infty} \frac{1}{1-e^{i2\pi\tilde{q}_n\beta}} \frac{1}{n\tilde{q}_{n+1}} e^{i2\pi\tilde{q}_n\theta}\right).$$

But we have

$$\frac{1}{\tilde{q}_n(\tilde{q}_n+\tilde{q}_{n+1})} \leq (-1)^n(\beta - \frac{\tilde{p}_n}{\tilde{q}_n}) \leq \frac{1}{\tilde{q}_n\tilde{q}_{n+1}},$$

hence

$$\frac{\tilde{q}_{n+1}}{2\pi} \leq \frac{1}{\left|1 - e^{i2\pi\tilde{q}_n\beta}\right|} \leq \frac{\tilde{q}_{n+1}}{2};$$

from the right hand side of this inequality we deduce that ψ is L^2. From the left hand side, it appears that the series is not absolutely convergent. Since it is a lacunar series (the q_n's increase exponentially), a theorem by Zygmund states that it is not continuous [5]. □

Finally, the last step of our construction is to let $\{T^t\}$ be the special flow constructed over T with the ceiling function φ (that depends only on the variables x and y). We recall rapidly the definition: The flow $\{T^t\}$ is obtained by inducing on $\mathbf{T}^4 \times \mathbf{R}/\sim$, where $\sim$ is the identification $(x, y, \theta, z, s + \varphi(x, y)) \sim (T(x, y, \theta, z), s)$, the action

$$\begin{array}{rcl} \mathbf{T}^4 \times \mathbf{R} & \rightarrow & \mathbf{T}^4 \times \mathbf{R} \\ (x, y, \theta, z, s) & \rightarrow & (x, y, \theta, z, s + t). \end{array}$$

The flow $\{T^t\}$, thus obtained, is analytic and preserves the normalized Lebesgue measure on $M_{T,\varphi} = \mathbf{T}^4 \times \mathbf{R}/\sim$, i.e. the product of the Haar measure on the basis $\mathbf{T}^4$ with the Lebesgue measure on the fibers. This is the flow we will work with and the theorem we want to prove is the following:

Theorem 1. *The flow $\{T^t\}$ is minimal and topologically mixing, and is not ergodic.*

First, the flow is minimal and nonergodic because T is minimal nonergodic. We only have to prove topological mixing.

In the sequel, we will use the following notations: By rectangle on $\mathbf{T}^2$ we designate a direct product of intervals of the circle. If $R \subset \mathbf{T}^2$ and $V \subset \mathbf{T}^2$ are such rectangles, $R \times V \times \{0\}$ designates a set of codimension 1 of the space $M_{T,\varphi}$ situated on the basis $\mathbf{T}^4$. In this expression, R encloses the coordinates x and y while V contains θ and z. By u we will denote a couple of coordinates (θ, z), and the variable r will be used to denote coordinates (x, y).

We will prove the following proposition that implies more than topological mixing:

Proposition 3. *Given R, R', $V \subset \mathbf{T}^2$ rectangles, and u a point of $\mathbf{T}^2$; then there exists t_0 such that, for any $t \geq t_0$*

$$T^t\left(R \times \{u\} \times \{0\}\right) \bigcap R' \times V \times \{0\} \neq \emptyset. \tag{5}$$

The sets involved in the proposition are taken to be on the basis $\mathbf{T}^4$. But the same equation (5) would clearly be true when t is large enough, for any couple of sets $T^s\,(R \times \{u\} \times \{0\})$ and $T^{s'}\left(R' \times V \times \{0\}\right)$, with $s, s' \in \mathbf{R}$. Since any two open sets of the space $M_{T,\varphi}$ contain sets of the precedent type, this proposition implies topological mixing for the flow.

Remark. We said that the property is stronger than topological mixing because the sets that intersect are respectively of dimension 2 and 4 in the five dimensional space where the flow acts.

The mechanism producing topological mixing is the following: because the Birkhoff sums of φ are *always* stretching when m is large (in one or in the other direction x and y); for large t, the image of $R \times \{u\} \times \{0\}$ by the flow at time t contains a union of almost vertical strips whose projection on the basis follows the trajectory under T of $R \times \{u\}$. So, by minimality of T one of the base points of these strips intersects the set $R' \times V \times \{0\}$. Since this is valid for all t large enough, topological mixing is proved. We go now to the detail of the proof.

Definition 1. For any $r \in \mathbf{T}^2$, and any positive time t, there is a unique integer m, such that

$$0 \leq t - \sum_{k=0}^{m-1} \varphi\left(R^k_{\alpha,\alpha'}(r)\right) < \varphi\left(R^m_{\alpha,\alpha'}(r)\right).$$

We denote this number m, by $N(r, t)$.
Note that because $\frac{1}{2} \leq \varphi \leq 2$, $N(r, t) \in [\frac{t}{2}, 2t]$ for any r. By definition of the special flow:

$$T^t(r, u, 0) = \left(R^{N(r,t)}_{\alpha,\alpha'}(r),\, F^{N(r,t)}(u), t - \varphi_{N(r,t)}(r)\right).$$

So, if m is such that $t - \varphi_m(r) = 0$, then $m = N(r, t)$ and

$$T^t(r, u, 0) = \left(R^m_{\alpha,\alpha'}(r),\, F^m(u), 0\right).$$

The stretch property of the Birkhoff sums of φ implies

Lemma 1 (Consequence of stretch). *Given a rectangle R, there exists t_0 such that for any $t \geq t_0$, we can find an $m_0 \geq \frac{t}{2}$ with the following property:*
For all $m \in [m_0, m_0 + {m_0}^{1/4}]$, there is an $r_m \in R$ such that

$$\varphi_m(r_m) = t.$$

Proof. We will assume t is in an interval of the type $[e^{2q_n}, e^{2q'_n}]$, for some integer n (the case $t \in [e^{2q'_n}, e^{2q_{n+1}}]$ being similar). By the Proposition 1 there is an interval $J = [j_1, j_2] \times \{y_0\}$ of R such that (3) holds on $J \times \{y_0\}$. Let $m_2 = N(j_2, y_0, t)$. In Definition 1, we saw that $m_2 \in [\frac{t}{2}, 2t] \subset [\frac{e^{2q_n}}{2}, 2e^{2q'_n}]$. By definition also we have

$$0 \leq t - \varphi_{m_2}(j_2, y_0) < \varphi\left(R^{m_2}_{\alpha,\alpha'}(j_2, y_0)\right) \leq \frac{3}{2}. \tag{6}$$

Now, because $\varphi \geq \frac{1}{2}$, we obtain from the right hand side in (6), for any $k \geq 3$

$$t - \varphi_{m_2+k}(j_2, y_0) < 0. \tag{7}$$

Next, if we look at the left extremity of $J \times \{y_0\}$, we have due to the left hand side in (6)

$$\begin{aligned} t - \varphi_{m_2}(j_1, y_0) &= t - \varphi_{m_2}(j_2, y_0) + \varphi_{m_2}(j_2, y_0) - \varphi_{m_2}(j_1, y_0) \\ &\geq \varphi_{m_2}(j_2, y_0) - \varphi_{m_2}(j_1, y_0) \\ &\geq \inf_{x \in J} \frac{\partial \varphi_{m_2}}{\partial x}(x, y_0)|J| \\ &\geq \frac{m_2}{q_n^2 e^{q_n}} \end{aligned}$$

from (3). Because $m_2 \geq \frac{e^{2q_n}}{2}$ the last inequality implies

$$t - \varphi_{m_2}(j_1, y_0) \geq 3m_2^{1/4},$$

since $\varphi \leq \frac{3}{2}$, we conclude that for any $k \leq 2{m_2}^{1/4}$

$$t - \varphi_{m_2+k}(j_1, y_0) \geq 0. \tag{8}$$

We take now $m_0 = m_2 + 3$ and we deduce Lemma 1 from (7) and (8) using the intermediate value theorem on the interval $J \times \{y_0\}$. □

The fact that the diffeomorphism T on $\mathbf{T}^4$ is minimal enables us to state the following lemma, the proof of which is direct by compacity:

Lemma 2 (Minimality). *Given two rectangles $\overline{R}'$ and V, and any point (r, u) in $\mathbf{T}^4$, there exists $A \in \mathbf{N}$ such that: For any $m_0 \geq A$, there exists $m \in [m_0, m_0 + {m_0}^{1/4}]$ satisfying*

$$\left(R^m_{\alpha,\alpha'}(r), F^m(u)\right) \in \overline{R}' \times V.$$

Now we prove Proposition 3:

Proof. We can assume that R is very small and take a rectangle $\overline{R'} \subset R'$ such that for any m, if an $r \in R$ satisfies

$$R^m_{\alpha,\alpha'}(r) \in \overline{R'},$$

then

$$R^m_{\alpha,\alpha'}(R) \subset R'.$$

Now, let A be the number given by Lemma 2 and assume $t \geq 2A$. By Lemma 1, there exists an $m_0 \geq \frac{t}{2} \geq A$ such that the set $T^t\,(R \times \{u\} \times \{0\})$ contains the points $\left(R^m_{\alpha,\alpha'}(r_m), F^m(u), 0\right)$ for *every* $m \in [m_0, m_0 + {m_0}^{1/4}]$. On the other hand by Lemma 2, applied to (r_0, u) where $r_0 \in R$ is arbitrarily chosen, we have for some $\check{m} \in [m_0, m_0 + {m_0}^{1/4}]$, that $\left(R^{\check{m}}_{\alpha,\alpha'}(r_0), F^{\check{m}}(u)\right) \in \overline{R}' \times V$. Hence $\left(R^{\check{m}}_{\alpha,\alpha'}(R), F^{\check{m}}(u)\right) \subset R' \times V$, and in particular $\left(R^{\check{m}}_{\alpha,\alpha'}(r_{\check{m}}), F^{\check{m}}(u)\right) \in R' \times V$.□

To conclude we want to derive from Theorem 1 the following

Theorem 2. *There exists an analytic diffeomorphism of* $\mathbf{T}^5$ *that preserve the Haar measure, that is minimal and topologically mixing, but not ergodic.*

Any time t_0 map of the flow we studied is conjugate to an analytic diffeomorphism of $\mathbf{T}^5$ that preserves the Haar measure. From Theorem 1 we have that T^{t_0} is topologically mixing and nonergodic. We can obtain Theorem 2 from Theorem 1 if we prove the following general fact

Proposition 4. *Let* $\{T^t\}$ *be a minimal flow on a compact metric space* M*, then for a dense* G_δ *set of* t *in* $\mathbf{R}$*, the time-*t *map of the flow is minimal.*

Proof. The proof we will give of this proposition is standard. We remind first the definitions: A flow $\{T^t\}$ on M is minimal if and only if the only closed sets $X \subset M$ such that

$$T^t(X) = X, \ \ for\ all\ t \in \mathbf{R}_+,$$

are M or the empty set $\emptyset$. A diffeomorphism T of M is minimal if and only if the only closed sets $X \subset M$ such that

$$T(X) = X,$$

are M or the empty set $\emptyset$.

Assume now $\{T^t\}$ is a minimal flow on a compact metric space M. Clearly, the flow is transitive, i.e. for any open sets $\mathcal{O}$ and $\mathcal{V}$ of M we have

$$\bigcup_{t \in \mathbf{R}_+} T^t(\mathcal{O}) \bigcap \mathcal{V} \neq \emptyset.$$

We will first show that for a dense G_δ set of $t \in \mathbf{R}$, the time-t map of the flow is transitive. Let $\{\mathcal{O}_i\}$ be a countable basis of open sets of M. Define

$$T_{i,j,m} = \{t \in \mathbf{R} \,/\, \bigcup_{k \geq m} T^{kt}(\mathcal{O}_i) \bigcap \mathcal{O}_j = \emptyset\}.$$

The set $T_{i,j,m}$ is closed and has an empty interior: Indeed, if $t \in T_{i,j,m}$, then $pt \in T_{i,j,m}$ for any $p \in \mathbf{N}^*$; therefore if $T_{i,j,m}$ contains an interval it will contain $[a, +\infty[$ for some $a \in \mathbf{R}$ which obviously contradicts the transitivity of the flow. Besides $T_{i,j,m}$ is closed because its complement is clearly open.
The complement of $\bigcup_{i,j,m \in \mathbf{N}} T_{i,j,m}$ is exactly the set of times such that the map T^t is transitive. From what was underlined above it is a dense G_δ in $\mathbf{R}$.

Knowing that $\{T^t\}$ is in fact minimal we will show that: *the same parameters t for which T^t is transitive are such that T^t is minimal.*
Let $t_0 \in \mathbf{R}$ such that T^{t_0} is transitive and let $X \subset M$ be a closed nonempty set such that $T^{t_0}(X) = X$. For $t \in \mathbf{R}_+$, define

$$X_t = \bigcup_{s \in [0,t]} T^s(X).$$

The closed set X_{t_0} is invariant by the flow. Since the flow is minimal, we have $X_{t_0} = M$.
On the other hand, since for any $t \in \mathbf{R}$

$$T^{t_0}(X_t) = X_t,$$

we have by transitivity of T^{t_0} that either $X_t = M$ or X_t has an empty interior. In particular, for $n \in \mathbf{N}^*$, $X_{t_0/n}$ is either M or has an empty interior. Since $\bigcup_{k=0}^{n-1} T^{kt_0/n} X_{t_0/n} = X_{t_0} = M$, it follows that $X_{t_0/n} = M$. But this holds for every integer $n > 0$, hence $X = M$. □

Acknowledgments. I wish to thank Enrique Pujals for pointing out this question to me, and Patrice Le Calvez for a simplification of the original proof.

References

[1] B. R. Fayad. Analytic mixing reparametrizations of irrational flows on the torus $\mathbf{T}^n$, $n \geq 3$. *To appear in Ergodic Theory Dynamical Systems.*

[2] H. Furstenberg. Strict ergodicity and transformation of the torus. *Amer. J. Math.*, **83:** (1961), 573–601.

[3] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, chapter 4. Cambridge University Press, Cambridge, 1995.

[4] J-C. Yoccoz. *Petits diviseurs en dimension 1.* Astérisque (1982), Appendix 1.

[5] Zygmund. *Trigonometric series*, chapter VI, 6. Cambridge University Press, (1959).

Bassam R. Fayad
Centre de Mathématiques
UMR 7640 du CNRS,
École Polytechnique
91128 Palaiseau Cedex
France

E-mail: Fayadb@math.polytechnique.fr